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# DETERMINATION OF SECOND AND FOURTH ORDER SECTORIAL HARMONICS IN EARTH'S GRAVITY FIELD FROM THE MOTION OF TWELVE-HOUR SATELLITES

BY

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SUMMARY

Values of the $\overline{C}_2^{(2)}$, $\overline{S}_2^{(2)}$, $\overline{C}_4^{(4)}$, and $\overline{S}_4^{(4)}$ sectorial harmonics in the earth's gravity field are obtained from the motion of two twelve-hour satellites. Each satellite was studied for a period in excess of two hundred days.

## LIST OF SYMBOLS

| | |
|---|---|
| $\{a, e, I, M, \omega, \Omega\}$ | Keplerian elements of the satellite orbit |
| $a_e$ | mean radius of the earth |
| $\left(C_n^{(m)}, S_n^{(m)}\right)$ | unnormalized gravity harmonics of the earth |
| $\left(\bar{C}_n^{(m)}, \bar{S}_n^{(m)}\right)$ | normalized gravity harmonics of the earth |
| $f$ | true anomaly |
| $G$ | gravitational constant |
| $M_\oplus$ | mass of the earth |
| $n$ | mean motion of the satellite |
| $n'$ | speed of rotation of the earth |
| $P_n^{(m)}(x)$ | associated Legendre function |
| $R$ | "resonant" part of U |
| $(r, \beta, \lambda)$ | spherical coordinates |
| $U$ | potential function for the earth |
| $X_{q+n}^{-p,q}(e)$ | Hansen coefficient defined in Reference (3) |
| $\delta a$ | perturbation in semimajor axis |
| $\theta_g$ | Greenwich Sidereal Time |
| $\mu$ | $G M_\oplus$ |
| $\dot{\Omega}$ | secular motion of the ascending node |
| $\dot{\omega}$ | secular motion of the argument of perigee |

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# DETERMINATION OF SECOND AND FOURTH ORDER SECTORIAL HARMONICS IN EARTH'S GRAVITY FIELD FROM THE MOTION OF TWELVE-HOUR SATELLITES

## INTRODUCTION

Several determinations of the tesseral harmonic coefficients have been carried out since the dawn of the space age. In many of these works the harmonics had to be derived from small short period perturbations in the motion of close satellites. A particular pair or set of pairs of gravity harmonics $\left(\bar{C}_n^{(m)}, \bar{S}_n^{(m)}\right)$ are more readily obtained if the mean motion of the satellite and speed of the rotation of the earth are commensurable. In this case both the period and amplitude of the perturbations are increased. Further, if the semimajor axis of the orbit is large, the effects of the harmonics are more easily separated according to degree. For both satellites 1964 49D (Cosmos 41) and 1965 30A (Molniya 1) the semimajor axis is greater than four earth radii and therefore permits such separation. The effect of the lower degree harmonics are further separated due to the presence of a nearly two to one commensurability in the mean motion and speed of rotation of the earth. This separation occurs in only certain of the second, third, fourth and fifth degree terms due to the larger amplitude and period of these terms. Satellites for which this phenomenon occurs are called twelve-hour satellites since they make two circuits in one day.

## THE DISTURBING FUNCTION

The recommended form for the potential of the earth as given in Reference (1) is

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{r}\right)^n P_n^{(m)}(\sin\beta)\left(C_n^{(m)}\cos m\lambda + S_n^{(m)}\sin m\lambda\right)\right] \tag{1}$$

where

$\mu$ is the product of the Gravitational Constant G and the mass of the earth $M_\oplus$.

and where

$a_e$ is the mean equatorial radius of the earth

r is the distance from the center of the earth

$\beta$ is the latitude

$\lambda$ is the longitude

$P_n^{(m)}$ is the associated Lengendre polynomial defined by

$$P_n^{(m)}(x) = \frac{1}{2^n\, n!}\left(1 - x^2\right)^{m/2} \frac{d^{n+m}\left(x^2 - 1\right)^n}{dx^{n+m}} \quad . \tag{2}$$

The relationship between the spherical coordinates $(r, \beta, \lambda)$ and the Keplerian elements is

$$\sin\beta = \sin I \sin(f + \omega)$$

$$\cos\beta\cos\lambda = \frac{1}{2}\left[(1 + \cos I)\cos\left(f + \omega + \Omega - \theta_g\right) + (1 - \cos I)\cos\left(f + \omega - \Omega + \theta_g\right)\right]$$

$$\cos\beta\sin\lambda = \frac{1}{2}\left[(1 + \cos I)\cos\left(f + \omega + \Omega - \theta_g\right) - (1 - \cos I)\cos\left(f + \omega - \Omega + \theta_g\right)\right] \tag{3}$$

where f is the true anomaly and $\theta_g$ is the Greenwich sidereal time.

The portion of the potential of the earth that is dependent upon the spherical harmonics of the earth is designated as the disturbing function, R. Therefore,

$$R = U - \frac{\mu}{r} \tag{4}$$

After substituting Equations (3) into Equation (1) and making use of Equation (4), the disturbing function to be used in this paper is

$$
\begin{aligned}
R = \; & \frac{3}{4}\frac{\mu a_e^{\,2}}{a^3}\left(\frac{a}{r}\right)^3 \Big\{(1+\cos I)^2\Big[C_2^{(2)}\cos\left(2f+2\omega+2\Omega-2\theta_g\right)+S_2^{(2)}\sin\left(2f\right. \\
& \left.+2\omega+2\Omega-2\theta_g\right)\Big]+2\sin^2 I\Big[C_2^{(2)}\cos\left(2\Omega-2\theta_g\right)+S_2^{(2)}\sin\left(2\Omega-2\theta_g\right)\Big] \\
& +(1-\cos I)^2\Big[C_2^{(2)}\cos\left(2f+2\omega-2\Omega+2\theta_g\right)+S_2^{(2)}\sin\left(2f+2\omega-2\Omega+2\theta_g\right)\Big]\Big\} \\
& +\frac{15}{8}\frac{\mu a_e^{\,3}}{a^4}\left(\frac{a}{r}\right)^4 \sin I\;\Big\{(1+\cos I)^2\Big[C_3^{(2)}\sin\left(3f+3\omega+2\Omega-2\theta_g\right)-S_3^{(2)}\cos\left(3f\right. \\
& \left.+3\omega+2\Omega-2\theta_g\right)\Big]+(1+\cos I)(1-3\cos I)\Big[C_3^{(2)}\sin\left(f+\omega+2\Omega-2\theta_g\right) \\
& -S_3^{(2)}\cos\left(f+\omega+2\Omega-2\theta_g\right)\Big]+(1-\cos I)(1+3\cos I)\Big[C_3^{(2)}\sin\left(f+\omega-2\Omega+2\theta_g\right) \\
& -S_3^{(2)}\cos\left(f+\omega-2\Omega+2\theta_g\right)\Big]+(1-\cos I)^2\Big[C_3^{(2)}\sin\left(3f+3\omega-2\Omega+2\theta_g\right) \\
& -S_3^{(2)}\cos\left(3f+3\omega-2\Omega+2\theta_g\right)\Big]\Big\} \\
& -\frac{15}{32}\frac{\mu a_e^{\,4}}{a^5}\left(\frac{a}{r}\right)^5 \Big\{7\sin^2 I\,(1+\cos I)^2\Big[C_4^{(2)}\cos\left(4f+4\omega+2\Omega-2\theta_g\right)+S_4^{(2)}\sin\left(4f\right. \\
& \left.+4\omega+2\Omega-2\theta_g\right)\Big]+4(1+\cos I)^2\left(1-7\cos I+7\cos^2 I\right)\Big[C_4^{(2)}\cos\left(2f+2\omega+2\Omega\right. \\
& \left.-2\theta_g\right)+S_4^{(2)}\sin\left(2f+2\omega+2\Omega-2\theta_g\right)\Big]-6\sin^2 I\Big[C_4^{(2)}\cos\left(2\Omega-2\theta_g\right)+S_4^{(2)}\sin\left(2\Omega\right. \\
& \left.-2\theta_g\right)\Big]+4(1-\cos I)^2\left(1+7\cos I+7\cos^2 I\right)\Big[C_4^{(2)}\cos\left(2f+2\omega-2\Omega+2\theta_g\right) \\
& +S_4^{(2)}\sin\left(2f+2\omega-2\Omega+2\theta_g\right)\Big]+7\sin^2 I\,(1-\cos I)^2\Big[C_4^{(2)}\cos\left(4f+4\omega-2\Omega\right. \\
& \left.+2\theta_g\right)+S_4^{(2)}\sin\left(4f+4\omega-2\Omega+2\theta_g\right)\Big]\Big\}
\end{aligned}
$$

$$+\frac{105}{16}\,\frac{\mu a_e^{\,4}}{a^5}\left(\frac{a}{r}\right)^5 \Big\{(1+\cos I)^4\Big[C_4^{(4)}\cos\big(4f+4\omega+4\Omega-4\theta_g\big)+S_4^{(4)}\sin\big(4f+4\omega$$

$$+4\Omega-4\theta_g\big)\Big]+4\sin^2 I\,(1+\cos I)^2\Big[C_4^{(4)}\cos\big(2f+2\omega+4\Omega-4\theta_g\big)+S_4^{(4)}\sin\big(2f$$

$$+2\omega+4\Omega-4\theta_g\big)\Big]+6\sin^4 I\,\Big[C_4^{(4)}\cos\big(4\Omega-4\theta_g\big)+S_4^{(4)}\sin\big(4\Omega-4\theta_g\big)\Big]+4\sin^2 I\,(1$$

$$-\cos I)^2\Big[C_4^{(4)}\cos\big(2f+2\omega-4\Omega+4\theta_g\big)+S_4^{(4)}\sin\big(2f+2\omega-4\Omega+4\theta_g\big)\Big]+(1$$

$$-\cos I)^4\Big[C_4^{(4)}\cos\big(4f+4\omega-4\Omega+4\theta_g\big)+S_4^{(4)}\sin\big(4f+4\omega-4\Omega+4\theta_g\big)\Big]\Big\}$$

$$+\frac{105}{64}\,\frac{\mu a_e^{\,5}}{a^6}\left(\frac{a}{r}\right)^6\sin I\,\Big\{-3\sin^2 I\,(1+\cos I)^2\Big[C_5^{(2)}\sin\big(5f+5\omega+2\Omega-2\theta_g\big)$$

$$-S_5^{(2)}\cos\big(5f+5\omega+2\Omega-2\theta_g\big)\Big]-(1+\cos I)^2\big(1-12\cos I+15\cos^2 I\big)\Big[C_5^{(2)}\sin\ 3f$$

$$+3\omega+2\Omega-2\theta_g\big)-S_5^{(2)}\cos\big(3f+3\omega+2\Omega-2\theta_g\big)\Big]+2(1+\cos I)\,\big(1-3\cos I$$

$$-9\cos^2 I+15\cos^3 I\big)\,\Big[C_5^{(2)}\sin\big(f+\omega+2\Omega-2\theta_g\big)-S_5^{(2)}\cos\big(f+\omega+2\Omega-2\theta_g\big)\Big]$$

$$-2(1-\cos I)\,\big(1+3\cos I-9\cos^2 I-15\cos^3 I\big)\Big[C_5^{(2)}\sin\big(f+\omega-2\Omega+2\theta_g\big)$$

$$-S_5^{(2)}\cos\big(f+\omega-2\Omega+2\theta_g\big)\Big]+(1-\cos I)^2\,\big(1+12\cos I+15\cos^2 I\big)\Big[C_5^{(2)}\sin\big(3f$$

$$+3\omega-2\Omega+2\theta_g\big)-S_5^{(2)}\cos\big(3f+3\omega-2\Omega+2\theta_g\big)\Big]+3\sin^2 I\,(1-\cos I)^2\Big[C_5^{(2)}\sin\big(5f$$

$$+5\omega-2\Omega+2\theta_g\big)-S_5^{(2)}\cos\big(5f+5\omega-2\Omega+2\theta_g\big)\Big]\Big\}$$

$$+\frac{945}{32}\,\frac{\mu a_e^{\,5}}{a^6}\left(\frac{a}{r}\right)^6\sin I\,\Big\{(1+\cos I)^4\Big[C_5^{(4)}\sin\big(5f+5\omega+4\Omega-4\theta_g\big)-S_5^{(4)}\cos\big(5f+5\omega$$

$$+4\Omega-4\theta_g\big)\Big]+(1+\cos I)^2\,(3-5\cos I)\,\Big[C_5^{(4)}\sin\big(3f+3\omega+4\Omega-4\theta_g\big)-S_5^{(4)}\cos\big(3f$$

$$+3\omega+4\Omega-4\theta_g\big)\Big]+2\sin^2 I\,(1+\cos I)\,(1-5\cos I)\Big[C_5^{(4)}\sin\big(f+\omega+4\Omega-4\theta_g\big)$$

$$-S_5^{(4)}\cos\big(f+\omega+4\Omega-4\theta_g\big)\Big]-2\sin^2 I\,(1-\cos I)\,(1+5\cos I)\,\Big[C_5^{(4)}\sin\big(f+\omega$$

$$-4\Omega+4\theta_g\big)-S_5^{(4)}\cos\big(f+\omega-4\Omega+4\theta_g\big)\Big]-(1-\cos I)^3\,(3+5\cos I)\,\Big[C_5^{(4)}\sin\big(3f+3\omega$$

$$- 4\Omega + 4\theta_g\Big) - S_5^{(4)} \cos\Big(3f + 3\omega - 4\Omega + 4\theta_g\Big)\Big] - (1 - \cos I)^4 \Big[C_5^{(4)} \sin\Big(5f$$

$$+ 5\omega - 4\Omega + 4\theta_g\Big) - S_5^{(4)} \cos\Big(5f + 5\omega - 4\Omega + 4\theta_g\Big)\Big]\Big\} \tag{5}$$

It will be more convenient to have R expressed in terms of the mean anomaly rather than the true anomaly. If the eccentricity of the satellite were small enough, it would be possible to transform from the true anomaly to the mean anomaly by means of expressions found in Cayley's Tables, Reference (2). However, since the satellites that will be considered in this paper have eccentricities in excess of .7 and since the expressions in Cayley's Tables are carried out only to the seventh power in eccentricity, an alternative approach must be taken. A more desirable representation is obtained by making use of the following relationship given in Reference (3),

$$\left(\frac{a}{r}\right)^p \begin{matrix}\sin\\ \cos\end{matrix}\, qf = \sum_{n=-\infty}^{\infty} X_{q+n}^{-p,q}(e) \begin{matrix}\sin\\ \cos\end{matrix}\,(q+n)M \tag{6}$$

The quantity $X_{q+n}^{-p,q}(e)$ is the Hansen coefficient and is defined in Reference (3). The symbol M denotes the mean anomaly.

## THE EQUATION FOR THE SEMIMAJOR AXIS

The semimajor axis of an artificial satellite is almost always free of long period perturbations. However, they can arise from two sources. The accumulative effect of radiation pressure due to passage through the shadow, see References (4) and (5), and a resonance condition in the longitude dependent terms in the earth's potential can cause such variations. While the first of these effects is negligible for the satellites to be considered here, the second effect produces perturbations of periods in the neighborhood of 116 days and 58 days for 1964 49D and in the neighborhood of 144 days and 72 days for 1965 30A when one considers harmonics through the fifth degree, fourth order. These predictable periods were observed in the data.

The equation for the semimajor axis is now obtained to use in the determination of some of the gravity harmonics. The rate of change of the semimajor axis is

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial M} \quad . \tag{7}$$

After substituting Equation (6) into (7) and singling out the resonant terms, taking the partial derivative and integrating, the equation for the semimajor axis is obtained. It is

$$a = a_0 + \delta a \tag{8}$$

where,

$$\begin{aligned}
\delta a = {} & \alpha_1 \left[C_2^{(2)} \cos\left(M + 2\omega + 2\Omega - 2\theta_g\right) + S_2^{(2)} \sin\left(M + 2\omega + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_2 \left[C_2^{(2)} \cos\left(M + 2\Omega - 2\theta_g\right) + S_2^{(2)} \sin\left(M + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_3 \left[C_2^{(2)} \cos\left(M - 2\omega + 2\Omega - 2\theta_g\right) + S_2^{(2)} \sin\left(M - 2\omega + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_4 \left[C_3^{(2)} \sin\left(M + 3\omega + 2\Omega - 2\theta_g\right) - S_3^{(2)} \cos\left(M + 3\omega + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_5 \left[C_3^{(2)} \sin\left(M + \omega + 2\Omega - 2\theta_g\right) - S_3^{(2)} \cos\left(M + \omega + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_6 \left[C_3^{(2)} \sin\left(M - \omega + 2\Omega - 2\theta_g\right) - S_3^{(2)} \cos\left(M - \omega + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_7 \left[C_3^{(2)} \sin\left(M - 3\omega + 2\Omega - 2\theta_g\right) - S_3^{(2)} \cos\left(M - 3\omega + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_8 \left[C_4^{(2)} \cos\left(M + 4\omega + 2\Omega - 2\theta_g\right) + S_4^{(2)} \sin\left(M + 4\omega + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_9 \left[C_4^{(2)} \cos\left(M + 2\omega + 2\Omega - 2\theta_g\right) + S_4^{(2)} \sin\left(M + 2\omega + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_{10} \left[C_4^{(2)} \cos\left(M + 2\Omega - 2\theta_g\right) + S_4^{(2)} \sin\left(M + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_{11} \left[C_4^{(2)} \cos\left(M - 2\omega + 2\Omega - 2\theta_g\right) + S_4^{(2)} \sin\left(M - 2\omega + 2\Omega - 2\theta_g\right)\right] \\
& + \alpha_{12} \left[C_4^{(2)} \cos\left(M - 4\omega + 2\Omega - 2\theta_g\right) + S_4^{(2)} \sin\left(M - 4\omega + 2\Omega - 2\theta_g\right)\right]
\end{aligned}$$

$$+\alpha_{13}\left[C_4^{(4)}\cos\left(2M+4\omega+4\Omega-4\theta_g\right)+S_4^{(4)}\sin\left(2M+4\omega+4\Omega-4\theta_g\right)\right]$$

$$+\alpha_{14}\left[C_4^{(4)}\cos\left(2M+2\omega+4\Omega-4\theta_g\right)+S_4^{(4)}\sin\left(2M+2\omega+4\Omega-4\theta_g\right)\right]$$

$$+\alpha_{15}\left[C_4^{(4)}\cos\left(2M+4\Omega-4\theta_g\right)+S_4^{(4)}\sin\left(2M+4\Omega-4\theta_g\right)\right]$$

$$+\alpha_{16}\left[C_4^{(4)}\cos\left(2M-2\omega+4\Omega-4\theta_g\right)+S_4^{(4)}\sin\left(2M-2\omega+4\Omega-4\theta_g\right)\right]$$

$$+\alpha_{17}\left[C_4^{(4)}\cos\left(2M-4\omega+4\Omega-4\theta_g\right)+S_4^{(4)}\sin\left(2M-4\omega+4\Omega-4\theta_g\right)\right]$$

$$+\alpha_{18}\left[C_5^{(2)}\sin\left(M+5\omega+2\Omega-2\theta_g\right)-S_5^{(2)}\cos\left(M+5\omega+2\Omega-2\theta_g\right)\right]$$

$$+\alpha_{19}\left[C_5^{(2)}\sin\left(M+3\omega+2\Omega-2\theta_g\right)-S_5^{(2)}\cos\left(M+3\omega+2\Omega-2\theta_g\right)\right]$$

$$+\alpha_{20}\left[C_5^{(2)}\sin\left(M+\omega+2\Omega-2\theta_g\right)-S_5^{(2)}\cos\left(M+\omega+2\Omega-2\theta_g\right)\right]$$

$$+\alpha_{21}\left[C_5^{(2)}\sin\left(M-\omega+2\Omega-2\theta_g\right)-S_5^{(2)}\cos\left(M-\omega+2\Omega-2\theta_g\right)\right]$$

$$+\alpha_{22}\left[C_5^{(2)}\sin\left(M-3\omega+2\Omega-2\theta_g\right)-S_5^{(2)}\cos\left(M-3\omega+2\Omega-2\theta_g\right)\right]$$

$$+\alpha_{23}\left[C_5^{(2)}\sin\left(M-5\omega+2\Omega-2\theta_g\right)-S_5^{(2)}\cos\left(M-5\omega+2\Omega-2\theta_g\right)\right]$$

$$+\alpha_{24}\left[C_5^{(4)}\sin\left(2M+5\omega+4\Omega-4\theta_g\right)-S_5^{(4)}\cos\left(2M+5\omega+4\Omega-4\theta_g\right)\right]$$

$$+\alpha_{25}\left[C_5^{(4)}\sin\left(2M+3\omega+4\Omega-4\theta_g\right)-S_5^{(4)}\cos\left(2M+3\omega+4\Omega-4\theta_g\right)\right]$$

$$+\alpha_{26}\left[C_5^{(4)}\sin\left(2M+\omega+4\Omega-4\theta_g\right)-S_5^{(4)}\cos\left(2M+\omega+4\Omega-4\theta_g\right)\right]$$

$$+\alpha_{27}\left[C_5^{(4)}\sin\left(2M-\omega+4\Omega-4\theta_g\right)-S_5^{(4)}\cos\left(2M-\omega+4\Omega-4\theta_g\right)\right]$$

$$+ \alpha_{28} \left[C_5^{(4)} \sin\left(2M - 3\omega + 4\Omega - 4\theta_g\right) - S_5^{(4)} \cos\left(2M - 3\omega + 4\Omega - 4\theta_g\right)\right]$$

$$+ \alpha_{29} \left[C_5^{(4)} \sin\left(2M - 5\omega + 4\Omega - 4\theta_g\right) - S_5^{(4)} \cos\left(2M - 5\omega + 4\Omega - 4\theta_g\right)\right] \quad (9)$$

and where

$$\alpha_1 = \frac{3\, n a_e^2 (1 + \cos I)^2 X_1^{-3,2}}{2a(n + 2\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_2 = \frac{3\, n a_e^2 \sin^2 I \, X_1^{-3,0}}{a(n + 2\dot{\Omega} - 2n')}$$

$$\alpha_3 = \frac{3\, n a_e^2 (1 - \cos I)^2 X_1^{-3,-2}}{2a(n - 2\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_4 = \frac{15\, n a_e^3 \sin I \,(1 + \cos I)^2 X_1^{-4,3}}{4a^2 (n + 3\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_5 = \frac{15\, n a_e^3 \sin I \,(1 + \cos I)(1 - 3\cos I) X_1^{-4,1}}{4a^2 (n + \dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_6 = \frac{-15\, n a_e^3 \sin I \,(1 - \cos I)(1 + 3\cos I) X_1^{-4,-1}}{4a^2 (n - \dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_7 = \frac{-15\, n a_e^3 \sin I \,(1 - \cos I)^2 X_1^{-4,-3}}{4a^2 (n - 3\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_8 = \frac{-105\, n a_e^4 \sin^2 I \,(1 + \cos I)^2 X_1^{-5,4}}{16a^3 (n + 4\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_9 = \frac{-15\, n a_e^4 (1 + \cos I)^2 (1 - 7\cos I + 7\cos^2 I) X_1^{-5,2}}{4a^3 (n + 2\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{10} = \frac{45\, n a_e^4 \sin^2 I \,(1 - 7\cos^2 I)\, X_1^{-5,0}}{8a^3\,(n + 2\dot{\Omega} - 2n')}$$

$$\alpha_{11} = \frac{-15\, n a_e^4 (1 - \cos I)^2\,(1 + 7\cos I + 7\cos^2 I)\, X_1^{-5,-2}}{4a^3\,(n - 2\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{12} = \frac{-105\, n a_e^4 \sin^2 I\,(1 - \cos I)^2\, X_1^{-5,-4}}{16a^3\,(n - 4\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{13} = \frac{105\, n a_e^4\,(1 + \cos I)^4\, X_2^{-5,4}}{8a^3\,(n + 2\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{14} = \frac{105\, n a_e^4 \sin^2 I\,(1 + \cos I)^2\, X_2^{-5,2}}{2a^3\,(n + \dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{15} = \frac{315\, n a_e^4 \sin^4 I\, X_2^{-5,0}}{4a^3\,(n + 2\dot{\Omega} - 2n')}$$

$$\alpha_{16} = \frac{105\, n a_e^4 \sin^2 I\,(1 - \cos I)^2\, X_2^{-5,-2}}{2a^3\,(n - \dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{17} = \frac{105\, n a_e^4\,(1 - \cos I)^4\, X_2^{-5,-4}}{8a^3\,(n - 2\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{18} = \frac{-315\, n a_e^5 \sin^3 I\,(1 + \cos I)^2\, X_1^{-6,5}}{32a^4\,(n + 5\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{19} = \frac{-105\, n a_e^5 \sin I\,(1 + \cos I)^2\,(1 - 12\cos I + 15\cos^2 I)\, X_1^{-6,3}}{32a^4\,(n + 3\dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{20} = \frac{105\, n a_e^5 \sin I\,(1 + \cos I)\,(1 - 3\cos I - 9\cos^2 I + 15\cos^3 I)\, X_1^{-6,2}}{16a^4\,(n + \dot{\omega} + 2\dot{\Omega} - 2n')}$$

$$\alpha_{21} = \frac{-105\, n a_e^5 \sin I\,(1-\cos I)(1+3\cos I-9\cos^2 I-15\cos^3 I)\,X_1^{-6,-1}}{16a^4\,(n-\dot{\omega}+2\dot{\Omega}-2n')}$$

$$\alpha_{22} = \frac{105\, n a_e^5 \sin I\,(1-\cos I)^2(1+12\cos I+15\cos^2 I)\,X_1^{-6,-3}}{32a^4\,(n-3\dot{\omega}+2\dot{\Omega}-2n')}$$

$$\alpha_{23} = \frac{315\, n a_e^5 \sin^3 I\,(1-\cos I)^2\,X_1^{-6,-5}}{32a^4\,(n-5\dot{\omega}+2\dot{\Omega}-2n')}$$

$$\alpha_{24} = \frac{945\, n a_e^5 \sin I\,(1+\cos I)^4\,X_2^{-6,5}}{8a^4\,(2n+5\dot{\omega}+4\dot{\Omega}-4n')}$$

$$\alpha_{25} = \frac{945\, n a_e^5 \sin I\,(1+\cos I)^3(3-5\cos I)\,X_2^{-6,3}}{8a^4\,(2n+3\dot{\omega}+4\dot{\Omega}-4n')}$$

$$\alpha_{26} = \frac{945\, n a_e^5 \sin^3 I\,(1+\cos I)(1-5\cos I)\,X_2^{-6,1}}{4a^4\,(2n+\dot{\omega}+4\dot{\Omega}-4n')}$$

$$\alpha_{27} = \frac{-945\, n a_e^5 \sin^3 I\,(1-\cos I)(1+5\cos I)\,X_2^{-6,-1}}{4a^4\,(2n-\dot{\omega}+4\dot{\Omega}-4n')}$$

$$\alpha_{28} = \frac{-945\, n a_e^5 \sin I\,(1-\cos I)^3(3+5\cos I)\,X_2^{-6,-3}}{8a^4\,(2n-3\dot{\omega}+4\dot{\Omega}-4n')}$$

$$\alpha_{29} = \frac{-945\, n a_e^5 \sin I\,(1-\cos I)^5\,X_2^{-6,-5}}{8a^4\,(2n-5\dot{\omega}+4\dot{\Omega}-4n')} \qquad (10)$$

The quantity n′ is the speed of rotation of the earth.

## NUMERICAL RESULTS

In Table 1 the various resonant arguments present in the equation for the semimajor axis are listed. The periods of those terms for each satellite are

Table 1

Period (in days) of Resonant Terms, $A_{\alpha,\beta} = \alpha\left[M + 2(\Omega - \theta_g)\right] + \beta\omega$

| | $\alpha, \beta$ | 1964 49D | 1965 30A |
|---|---|---|---|
| 1 | 1,-5 | 112.854 | 150.400 |
| 2 | 1,-4 | 113.774 | 149.331 |
| 3 | 1,-3 | 114.709 | 148.277 |
| 4 | 1,-2 | 115.660 | 147.238 |
| 5 | 1,-1 | 116.626 | 146.213 |
| 6 | 1,0 | 117.609 | 145.202 |
| 7 | 1,1 | 118.608 | 144.206 |
| 8 | 1,2 | 119.625 | 143.222 |
| 9 | 1,3 | 120.659 | 142.252 |
| 10 | 1,4 | 121.711 | 141.295 |
| 11 | 1,5 | 122.782 | 140.351 |
| 12 | 2,-5 | 57.591 | 73.878 |
| 13 | 2,-4 | 57.830 | 73.619 |
| 14 | 2,-3 | 58.070 | 73.362 |
| 15 | 2,-2 | 58.313 | 73.106 |
| 16 | 2,-1 | 58.558 | 72.853 |
| 17 | 2,0 | 58.804 | 72.601 |
| 18 | 2,1 | 59.053 | 72.351 |
| 19 | 2,2 | 59.304 | 72.103 |
| 20 | 2,3 | 59.557 | 71.856 |
| 21 | 2,4 | 59.812 | 71.611 |
| 22 | 2,5 | 60.070 | 71.368 |

also given. One or more of the pairs of harmonics $(C_2^{(2)}, S_2^{(2)})$, $(C_3^{(2)}, S_3^{(2)})$, $(C_4^{(2)}, S_4^{(2)})$ and $(C_5^{(2)}, S_5^{(2)})$ appears in the coefficient of a trigonometric term for each of the first eleven periods in this list, and one of the pairs $(C_4^{(4)}, S_4^{(4)})$ or $(C_5^{(4)}, S_5^{(4)})$ appears in the coefficient of a trigonometric term for each of the last eleven periods. Since the two groupings of terms have periods that are so close, it would be most difficult to separate the effects of each pair of harmonics. Instead, it was assumed the tesseral harmonics $(C_3^{(2)}, S_3^{(2)})$, $(C_4^{(2)}, S_4^{(2)})$, $(C_5^{(2)}, S_5^{(2)})$ and $(C_5^{(4)}, S_5^{(4)})$ were known while the sectorial harmonics $(C_2^{(2)}, S_2^{(2)})$ and $(C_4^{(4)}, S_4^{(4)})$ were to be obtained by least squares from the data. The results are presented in terms of the normalized sectorial harmonics $(\bar{C}_n^{(m)}, \bar{S}_n^{(m)})$. They are related to $(C_n^{(m)}, S_n^{(m)})$ by $(\bar{C}_n^{(m)}, \bar{S}_n^{(m)}) = N_n^{(m)} (C_n^{(m)}, S_n^{(m)})$ where $N_n^{(m)} = [2(2n+1)(n-m)/(n+m)!]^{-1/2}$ for $m \neq 0$.

Several solutions were obtained for each satellite. In the first solution it was assumed that the tesseral harmonics were zero. Then in the next five solutions they were given values obtained by previous investigators as follows. In the second solution values obtained by Anderle, Reference (6) for the tesseral harmonics were used. In the third solution values obtained by Kaula, Reference (7) for $(C_3^{(2)}, S_3^{(2)})$, $(C_4^{(2)}, S_4^{(2)})$ and $(C_5^{(2)}, S_5^{(2)})$ were used along with Gaposchkin's L3 solution, Reference (8) for $(C_5^{(4)}, S_5^{(4)})$ since this pair did not appear in Reference (7). In solutions four and five values obtained by Gaposchkin in the L1 and L3 solutions of Reference (8) were used. In the last solution values obtained by Köhnlein, Reference (9) for the tesseral harmonics were used. Values for the sectorial harmonics $(C_2^{(2)}, S_2^{(2)})$ and $(C_4^{(4)}, S_4^{(4)})$ were derived for each of these six cases. In Tables 2 and 3 the results are tabulated first for the satellite 1964-49D and then for 1965-30A. Values assumed to be known for the tesseral harmonics are also indicated in the tables.

In Figures (1) and (2) the data points computed by NORAD and issued by the Goddard Space Flight Center are plotted. The solid curves represent a computed semimajor axis based upon the results of solution 1 of Tables 2 and 3.

Table 2*

Second and Fourth Degree Sectorial Harmonics Derived From 1964 49D.

| | $\bar{C}_2^{(2)}$ | $\bar{S}_2^{(2)}$ | $\bar{C}_3^{(2)}$ | $\bar{S}_3^{(2)}$ | $\bar{C}_4^{(2)}$ | $\bar{S}_4^{(2)}$ | $\bar{C}_4^{(4)}$ | $\bar{S}_4^{(4)}$ | $\bar{C}_5^{(2)}$ | $\bar{S}_5^{(2)}$ | $\bar{C}_5^{(4)}$ | $\bar{S}_5^{(4)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.19 | -1.72 | O | O | O | O | .45 | -.65 | O | O | O | O |
| 2 | 1.82 | -1.63 | A | A | A | A | -.01 | -.68 | A | A | A | A |
| 3 | 1.88 | -1.62 | KA | KA | KA | KA | .21 | -.66 | KA | KA | G3 | G3 |
| 4 | 1.96 | -1.71 | G1 | G1 | G1 | G1 | .20 | -.68 | G1 | G1 | G1 | G1 |
| 5 | 1.95 | -1.69 | G3 | G3 | G3 | G3 | .21 | -.66 | G3 | G3 | G3 | G3 |
| 6 | 1.95 | -1.70 | KO | KO | KO | KO | .19 | -.67 | KO | KO | KO | KO |

*Multiply all coefficients by $10^{-6}$

Table 3*

Second and Fourth Degree Sectorial Harmonic Derived From 1965 30A.

| | $\bar{C}_2^{(2)}$ | $\bar{S}_2^{(2)}$ | $\bar{C}_3^{(2)}$ | $\bar{S}_3^{(2)}$ | $\bar{C}_4^{(2)}$ | $\bar{S}_4^{(2)}$ | $\bar{C}_4^{(4)}$ | $\bar{S}_4^{(4)}$ | $\bar{C}_5^{(2)}$ | $\bar{S}_5^{(2)}$ | $\bar{C}_5^{(4)}$ | $\bar{S}_5^{(4)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.42 | -1.74 | O | O | O | O | .38 | -.45 | O | O | O | O |
| 2 | 2.06 | -1.56 | A | A | A | A | .78 | -.58 | A | A | A | A |
| 3 | 2.13 | -1.55 | KA | KA | KA | KA | .59 | -.52 | KA | KA | G3 | G3 |
| 4 | 2.19 | -1.65 | G1 | G1 | G1 | G1 | .60 | -.51 | G1 | G1 | G1 | G1 |
| 5 | 2.19 | -1.64 | G3 | G3 | G3 | G3 | .59 | -.51 | G3 | G3 | G3 | G3 |
| 6 | 2.19 | -1.65 | KO | KO | KO | KO | .61 | -.52 | KO | KO | KO | KO |

*Multiply all coefficients by $10^{-6}$

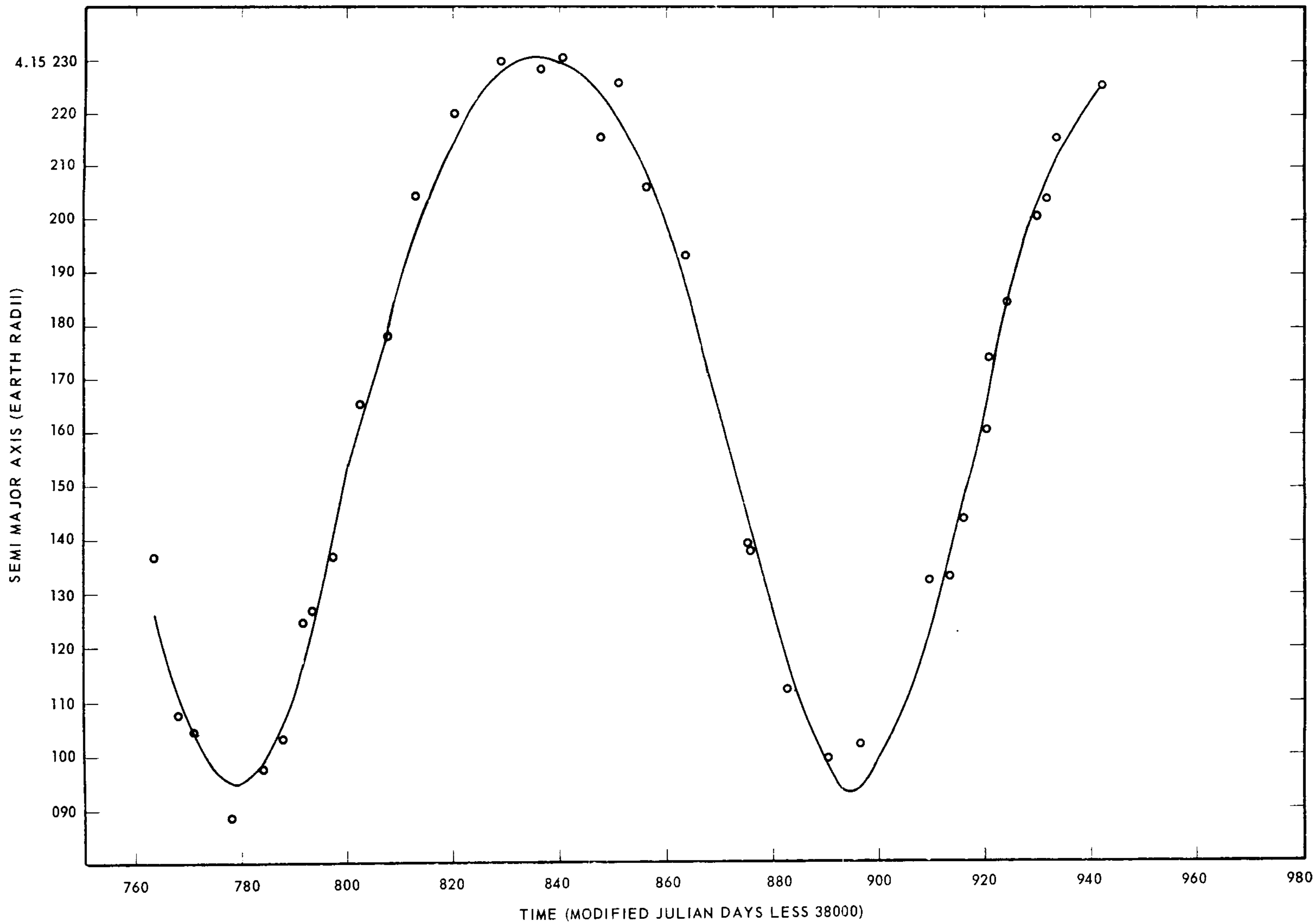


Figure 1 – Semimajor Axis of 1964 49D

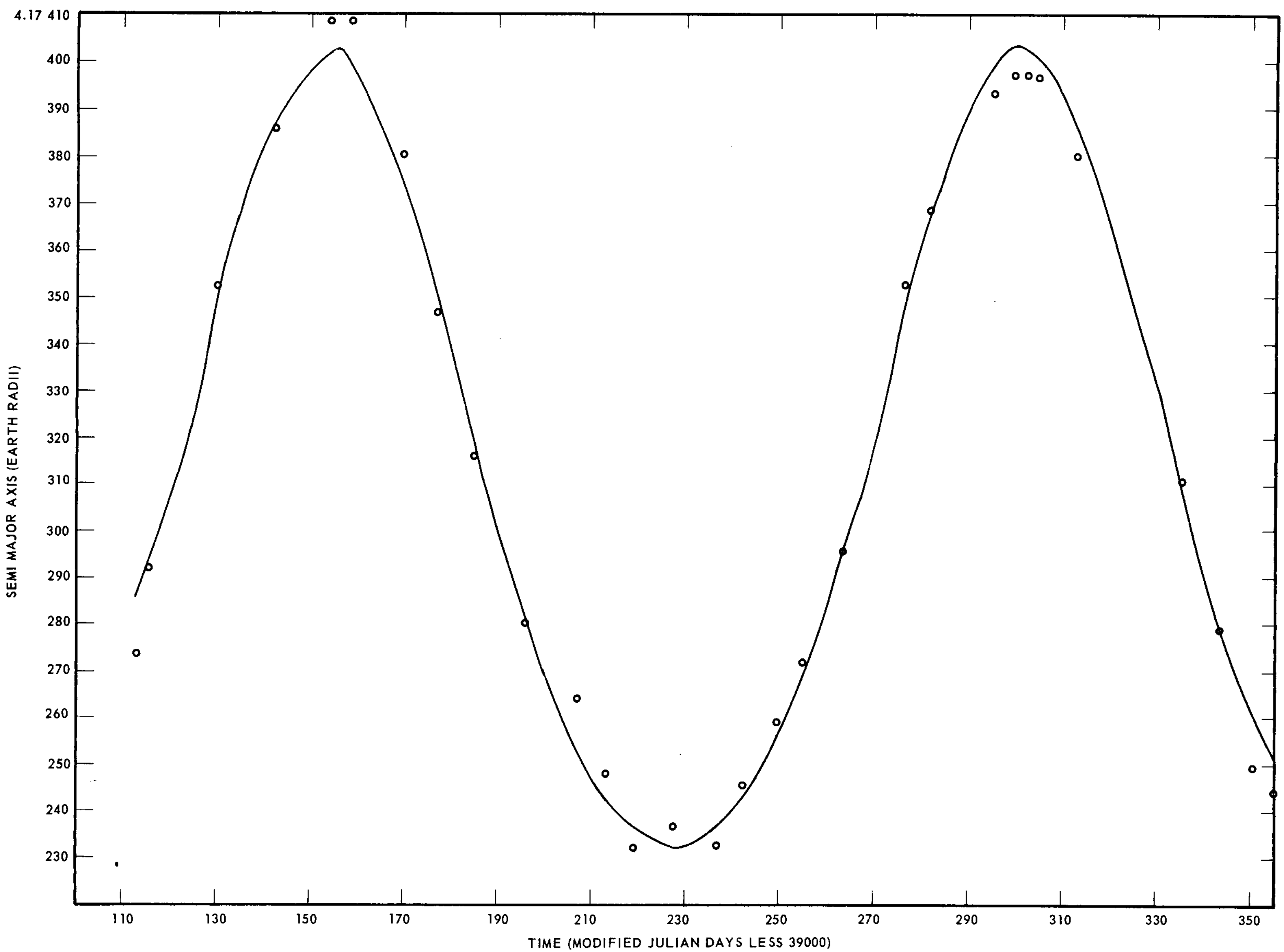


Figure 2 – Semimajor Axis of 1965 30A

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